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# DEFORMATION OF CERTAIN QUADRATIC ALGEBRAS AND THE CORRESPONDING QUANTUM SEMIGROUPS*

BY

J. DONIN AND S. SHNIDER

*Department of Mathematics, Bar-Ilan University*
*Ramat Gan 52900, Israel*
*e-mail: donin@macs.biu.ac.il shnider@macs.biu.ac.il*

ABSTRACT

Let $V$ be a finite-dimensional vector space. Given a decomposition $V \otimes V = \bigoplus_{i=1,\dots n} I_i$, define $n$ quadratic algebras $Q(V, J_{(m)})$ where $J_{(m)} = \bigoplus_{i \neq m} I_i$. There is also a quantum semigroup $M(V; I_1, \dots, I_n)$ which acts on all these quadratic algebras. The decomposition determines as well a family of associative subalgebras of $\mathrm{End}(V^{\otimes k})$, which we denote by $A_k = A_k(I_1, \dots, I_n)$, $k \geq 2$. In the classical case, when $V \otimes V$ decomposes into the symmetric and skewsymmetric tensors, $A_k$ coincides with the image of the representation of the group algebra of the symmetric group $S_k$ in $\mathrm{End}(V^{\otimes k})$. Let $I_{i,h}$ be deformations of the subspaces $I_i$. In this paper we give a criteria for flatness of the corresponding deformations of the quadratic algebras $Q(V, J_{(m),h})$ and the quantum semigroup $M(V; I_{1,h}, \dots, I_{n,h})$. It says that the deformations will be flat if the algebras $A_k(I_1, \dots, I_n)$ are semisimple and under the deformation their dimension does not change.

Usually, the decomposition into $I_i$ is defined by a given semisimple operator $S$ on $V \otimes V$, for which $I_i$ are its eigensubspaces, and the deformations $I_{i,h}$ are defined by a deformation $S_h$ of $S$. We consider the cases when $S_h$ is a deformation of Hecke or Birman–Wenzl symmetry, and also the case when $S_h$ is the Yang–Baxter operator which appears by a representation of the Drinfeld–Jimbo quantum group. Applying the flatness criteria we prove that in all these cases we obtain flat deformations of the quadratic algebras and the corresponding quantum semigroups.

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## 1. Introduction

Let $V$ be a finite-dimensional vector space over a field $\mathbf{k}$ of characteristic zero and $J$ a subspace of $V \otimes V$. We denote by $Q(V,J) = T(V)/\langle J\rangle$ the corresponding quadratic algebra, that is, the quotient of the tensor algebra, $T(V)$, by the two-sided ideal, $\langle J\rangle$, generated by $J$.

Suppose that we have a parametrized family of subspaces, $J_h$, depending analytically on $h$, then we can consider the parametrized family of quadratic algebras, $Q(V, J_h)$. This family is called "flat" if the dimension of each homogeneous component remains constant as $h$ varies. If the family is flat in a neighborhood of $h = h_0$ we say that $Q(V, J_h)$ is a "flat deformation" of $Q(V, J_{h_0})$.

We are interested in a class of quadratic algebras arising in the theory of quantum groups. In this case we fix a decomposition $V \otimes V = \bigoplus_{i=1,\dots,n} I_i$ into subspaces. Suppose deformations of these subspaces, $I_{i,h}$, $i = 1, \dots, n$, are given, thus we have the decomposition $V \otimes V = \bigoplus_{i=1,\dots,n} I_{i,h}$ for each $h$ from a neighborhood of zero in a linear space over $\mathbf{k}$, and $I_{i,0} = I_i$. In this setting we associate to the $n$-tuple $(I_{1,h}, \dots, I_{n,h})$ parametrized families of quadratic algebras, $Q(V, J_{(m),h})$, where $m = 1, \dots, n$, and a quantum semigroup $M_h = M(V; I_{1,h}, \dots, I_{n,h})$ (see Section 2) which acts on all the algebras $Q(V, J_{(m),h})$. Considered as an algebra, $M_h$ is a parametrized family of quadratic algebras as well.

We give a criterion for the flatness of these quadratic algebras. It is formulated in terms of subalgebras $A_k(I_{1,h}, \dots, I_{n,h}) \subset \mathrm{End}(V^{\otimes k})$ which are constructed as follows. Take a linear operator $B_h$ acting on $V \otimes V$ which has $n$ distinct eigenvalues with $I_{1,h}, \dots, I_{n,h}$ as the corresponding eigensubspaces. Then $A_k(I_{1,h}, \dots, I_{n,h})$ is generated by $B_{i,h} = \mathrm{id} \otimes \cdots \otimes B_h \otimes \cdots \otimes \mathrm{id}$ where $B_h$ appears in positions $i, i+1$. Another set of generators for this subalgebra is given by the projections $P^m_{i,h}$ onto the $m$-th eigenspace of $B_{i,h}$, thus $A_k(I_{1,h}, \dots, I_{n,h})$ really depends only on the decomposition of $V \otimes V$, but not on the choice of $B_h$.

Although the standard definitions of the quadratic algebras arising in the theory of quantum groups use the quantum permutation $S_h$, our criterion is formulated in terms of the subspace decomposition, because it is necessary to assume that the subspaces $I_i$, $i = 1, \dots, n$, for $h = 0$ can be defined as the limit of the eigensubspaces $I_{i,h}$ of $S_h$ for $h \neq 0$, even when this definition does not agree with the eigenspace decomposition for $S_0$ (see Example 3 in Section 4).

The basic criterion for the flatness is given in

THEOREM 1.1: *Let $V$ be a finite-dimensional vector space over $\mathbf{k}$. Suppose that $I_{i,h}$, $i = 1, \dots, n$, are deformations of $I_i$ depending analytically on $h$ in a neigh-*

*borhood of zero in a linear vector space over* $\mathbf{k}$, $I_{i,0} = I_i$. *Suppose that* $V^{\otimes 2} = \bigoplus_i^n I_{i,h}$ *is a direct sum decomposition for all* $h$. *Denote* $J_{(m),h} = \bigoplus_{i \neq m} I_{i,h}$. *Assume that for all* $k$,

(1) *the subalgebra* $A_k(I_1, \ldots, I_n)$ *of* $\mathrm{End}(V^{\otimes k})$ *is semisimple and*

(2) $\dim A_k(I_{1,h}, \ldots, I_{n,h})$ *does not depend on* $h$.

*Then all the quadratic algebra deformations* $Q(V, J_{(m),h})$ *of the quadratic algebras* $Q(V, J_{(m)})$ *are flat for all* $m$. *Moreover, the deformation* $M(V; I_{1,h}, \ldots, I_{n,h})$ *of the quantum semigroup* $M(V; I_1, \ldots, I_n)$ *is flat as well.*

The theorem can be reformulated for the case when formal deformations of subspaces are considered. In this case we suppose that $h$ is indeterminate.

THEOREM 1.2: *Let* $V$ *be a finite-dimensional vector space over* $\mathbf{k}$. *Suppose that* $I_{i,h}$, $i = 1, \ldots, n$, *are formal deformations of* $I_i$, *i.e.* $I_{i,h}$ *are* $\mathbf{k}[[h]]$ *submodules of* $V[[h]]$, $I_{i,0} = I_i$. *Suppose that* $V^{\otimes 2}[[h]] = \bigoplus_i^n I_{i,h}$ *is a direct sum decomposition. Denote* $J_{(m),h} = \bigoplus_{i \neq m} I_{i,h}$. *Assume that for all* $k$,

(1) *the subalgebra* $A_k(I_1, \ldots, I_n)$ *of* $\mathrm{End}(V^{\otimes k})$ *is semisimple and*

(2) *the subalgebras* $A_k(I_{1,h}, \ldots, I_{n,h})$ *are splitting* $\mathbf{k}[[h]]$ *submodules in* $\mathrm{End}(V^{\otimes k})[[h]]$, *i.e. there exists another submodule* $F_h$ *such that* $\mathrm{End}(V^{\otimes k})[[h]] = A_k(I_{1,h}, \ldots, I_{n,h}) \oplus F_h$.

*Then all the quadratic algebra deformations* $Q(V[[h]], J_{(m),h})$ *of the quadratic algebras* $Q(V, J_{(m)})$ *are free* $\mathbf{k}[[h]]$ *modules for all* $m$. *Moreover, the deformation* $M(V[[h]]; I_{1,h}, \ldots, I_{n,h})$ *of the quantum semigroup* $M(V[[h]]; I_1, \ldots, I_n)$ *is a free* $\mathbf{k}[[h]]$ *module as well.*

These theorems should be compared to a theorem of Drinfeld stating that if $Q(V, J)$ is a Koszul algebra, then a sufficient condition for flatness of $Q(V, J_h)$ is that the homogeneous components in degrees 2 and 3 have constant dimension (see Section 2). Our criterion replaces the Koszul condition on $Q(V, J)$ (which often is difficult to check) by the semisimplicity of $A_k(I_{1,0}, \ldots, I_{n,0})$. Even though there are an infinite number of conditions on constancy of dimension instead of the two in Drinfeld's criterion, in many examples the algebras $A_k(I_{1,h}, \ldots, I_{n,h})$ form families for which the constancy of dimension is proven uniformly for all $k$, making the criterion very effective. In Section 4 we demonstrate how this criterion can be applied to the quantum spaces and quantum semigroups defined by Faddeev, Reshetikhin, Takhtajan and Manin. In the papers [FRT] and [Ma] the question of flatness was not addressed. Some special cases were dealt with in [GGS1] and [Gu], but to the best of our knowledge, our result is the first that is generally applicable which does not use Koszul properties. In the cases of quantum spaces associated to the defining representation, $V$, of the classical simple Lie algebras, the algebras $A_k(I_{1,h},\ldots,I_{n,h})$ are either Hecke algebras or Birman–Wenzl algebras. We use some well known results on the structure of these algebras to establish the conditions of the theorem.

In an earlier paper, [DS], we used a direct calculation involving the Drinfeld associator to study the case when $S_h$ defining the decomposition of $V\otimes V$ arises from a representation of the Drinfeld–Jimbo quantum group. We also review these examples in light of the current method.

The paper is organized as follows. In Section 2 we introduce definitions and notations which are used in the paper. In Section 3 we prove some known facts on finite dimensional semisimple algebras and derive from them the criterion for the flatness. In Section 4 we give a number of examples which demonstrate that our criterion allows to prove flatness of quadratic algebras and quantum semigroups associated to almost all known quantum permutations.

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## 2. Quadratic algebras over k[[h]], some definitions and notations

Let $V$ be a module over a commutative ring $A$, $I$ a submodule of $V^{\otimes 2}=V\otimes V$ (all tensor products are considered over $A$). Denote by $I^{i,k}$ the submodule in $V^{\otimes k}$ of the form $V\otimes\cdots\otimes I\otimes\cdots\otimes V$ where $I$ occupies the positions $i,i+1$. Set $I^k=\sum_i I^{i,k}$ and $I^{(k)}=\bigcap_i I^{i,k}$. So, $I=I^2=I^{(2)}$.

We say that the ordered pair of submodules $(I,J)$, $I,J\subset V^{\otimes 2}$, is **well situated** if $V^{\otimes k}=I^{(k)}\oplus J^k$ for all $k\geq 2$. In particular, $V^{\otimes 2}=I\oplus J$.

For a submodule $J\in V^{\otimes 2}$ we denote by $Q_J=Q(V,J)$ the quadratic algebra $T(V)/\langle J\rangle$, where $T(V)$ is the tensor algebra over $V$ and $\langle J\rangle$ denotes the ideal generated by $J$. The algebra $Q_J$ is a graded one, its $k^{\text{th}}$ homogeneous component $Q_J^k$ is equal to $T^k(V)/J^k$ as $A$-module. If the pair $(I,J)$ is well situated the restriction of the natural mapping $T^k(V)\to Q_J^k$ gives an isomorphism $I^{(k)}\to Q_J^k$ of $A$-modules.

In the sequel we will deal with the cases when $A$ is either a field $\mathbf{k}$ of characteristic zero or the algebra $\mathbf{k}[[h]]$ of formal power series in a variable $h$. We will consider only modules of finite rank. Any free $\mathbf{k}[[h]]$-module of rank $n$ is isomorphic as $\mathbf{k}[[h]]$-module to $E\otimes_{\mathbf{k}}\mathbf{k}[[h]]=E[[h]]$ where $E$ is an $n$-dimensional vector space over $\mathbf{k}$, the module of formal power series in $h$ with coefficients from $E$.

We say that a submodule $J_h$ of a $\mathbf{k}[[h]]$-module $E_h$ is a **splitting submodule** if it has a complementary submodule $I_h$, i.e. $E_h = J_h \oplus I_h$. It is clear that when $E_h$ is a free module any submodule $J_h$ is free, but $J_h$ is a splitting one if and only if the module $E_h/J_h$ is free. We call a morphism of free modules, $\varphi$: $E_h \to V_h$, flat if Im$\varphi$ is a splitting submodule.

Let $J$ be a linear subspace in a vector space $E$ over $\mathbf{k}$. We say that $J_h$ is a **family of subspaces in** $E$, or a **(formal) deformation of the subspace** $J$, if $J_h$ is a splitting submodule in $E_h = E[[h]]$ such that $J_0 = J$. Here $J_0$ is the set of elements obtained from elements of $J_h$ replacing $h$ by 0. Note that for a submodule $J_h$ in $E[[h]]$ we have $\dim J_0 \leq \dim J_{h\neq 0}$, and $J_h$ defines a deformation of the subspace $J_0$ if $\dim J_0 = \dim J_{h\neq 0}$. Here $J_{h\neq 0}$ denotes the vector subspace in the "general" point, i.e. the vector subspace $J_h \otimes_{\mathbf{k}[[h]]} \mathbf{k}\{\{h\}\}$ in the vector space $V_h \otimes_{\mathbf{k}[[h]]} \mathbf{k}\{\{h\}\}$ over the field of formal Laurent series $k\{\{h\}\}$.

If $J_h$ is not a splitting submodule in $E[[h]]$, then the module $P_h = E[[h]]/J_h$ has a decomposition $P_h = P_h^{'} \oplus P_h^{''}$ where $P_h^{'}$ is a free module and $P_h^{''}$ is the torsion submodule of $P_h$, that is $b \in P_h^{''}$ if and only if there exists $m > 0$ such that $h^m b = 0$. Denote by $J_h^{'}$ the kernel of the natural projection $E[[h]] \to P_h^{'}$. It is clear that $J_h^{'}$ is a splitting submodule in $E[[h]]$ and $\dim J_{h\neq 0} = \dim J_{h\neq 0}^{'} = \dim J_0^{'}$. So we denote $J_0^{'}$ by $J_{h\to 0}$.

We will only consider quadratic $\mathbf{k}[[h]]$-algebras $Q(V_h, J_h)$ such that $V_h$ is a free module of finite rank and $J_h$ is a splitting submodule of $V_h^{\otimes 2}$. We associate to the quadratic algebra $Q(V_h, J_h)$ the quadratic algebra $Q(V, J)$ over $\mathbf{k}$ taking $V = V/hV_h$ and $J = J_h/hJ_h$ with the natural imbedding $J \to V^{\otimes 2}$. In this case we call the quadratic algebra $Q_h = Q(V_h, J_h)$ a deformation of the algebra $Q = Q(V, J)$. We call another deformation $Q(V_h^{'}, J_h^{'})$ of the algebra $Q(V, J)$ equivalent to $Q(V_h, J_h)$ if there exists an isomorphism $\phi$: $V_h^{'} \to V_h$ which induces the identity isomorphism on $V$ and the isomorphism $\phi \otimes \phi$: $V_h^{'} \otimes V_h^{'} \to V_h \otimes V_h$ gives an isomorphism $J_h^{'} \to J_h$. Since there exists an isomorphism $V[[h]] \to V_h$, any deformation $Q(V_h, J_h)$ of the algebra $Q(V, J)$ can be given by a formal deformation of the subspace $J$ in $V^{\otimes 2}$, i.e. is equivalent to a deformation of the form $Q(V[[h]], J_h^{'})$ where $J_0^{'} = J$.

Let $Q(V_h, J_h)$ be a quadratic algebra. Note that in general $J_h^k$ may not be a splitting submodule in $V_h^{\otimes k}$ for $k > 2$, so in this case the homogeneous component $Q_h^k = V_h^{\otimes k}/J_h^k$ will not be a free module. We call the deformation $Q_h$ a flat deformation (or quantization) of $Q$ if all modules $Q_h^k$ are free. Note that this terminology is not completely standard. For many authors the flatness condition is included in the definition of a deformation.

We mention here a theorem due to Drinfeld [Dr], which states that in case of the Koszul quadratic algebra $Q(V,J)$, [Ma], in order for all modules $Q_h^k$, $k>2$, to be free it is sufficient that the module $Q_h^3$ be free, i.e. the submodule $J_h^3$ be splitting in $V_h^{\otimes 3}$.

Let $V$ be a finite-dimensional vector space over $\mathbf{k}$. Fix subspaces $I_i$, $i=1,\dots,n$, in $V^{\otimes 2}$ such that $V^{\otimes 2}=\bigoplus_{i=1}^n I_i$. To such decomposition we associate the following quadratic algebras.

(i) For each $m=1,\dots,n$ set $Q_m=Q(V,J_m)=T(V)/\langle J_m\rangle$ where $J_{(m)}=\bigoplus_{i\neq m} I_i$, so $V^{\otimes 2}=I_m\oplus J_m$.

(ii) Define $M=M(V;I_1,\dots,I_n)$, a quantum semigroup, as follows. We identify $\mathrm{End}(V)=V\otimes V^*$ and put $M=(\mathrm{End}(V),I)$, the quadratic algebra where the subspace $I$ of $\mathrm{End}(V)^{\otimes 2}$ is defined as $I=\sigma_{2,3}(I_1\otimes I_1^{\perp}+\cdots+I_n\otimes I_n^{\perp})$. Here $I_k^{\perp}$ denotes the orthogonal to $I_k$ subspace in the dual space $V^*$ and $\sigma_{2,3}$ is the permutation of the second and third tensor components. The algebra $M$ has the natural bialgebra structure and the algebras $(V,J_i)$ make into comodules over $M$ [Ma].

The quantum semigroup $M$ also admits the description in the spirit of Faddeev–Reshetikhin–Takhtajan [FRT]. Let $\lambda_i$, $i=1,\dots,n$, be distinct elements from $\mathbf{k}$. Let $B$ be a linear operator acting on $V^{\otimes 2}$, which has $I_i$ as the eigensubspace corresponding to the eigenvalue $\lambda_i$ for all $i$. Identifying $\mathrm{End}(V^{\otimes 2})\cong\mathrm{End}(V)^{\otimes 2}$ via the Kronecker product we may view $B$ as an element of $\mathrm{End}(V)^{\otimes 2}$. Let $B=B_{(1)}\otimes B_{(2)}$ in the Sweedler notation. Then $I$ consists of the elements $X=X_{(1)}\otimes X_{(2)}$ of $\mathrm{End}(V)^{\otimes 2}$ having the form

$$BX-XB=B_{(1)}X_{(1)}\otimes B_{(2)}X_{(2)}-X_{(1)}B_{(1)}\otimes X_{(2)}B_{(2)}.$$

This means that coaction of $M$ on $V$ preserves all the subspaces $I_i$ of $V\otimes V$.

Suppose that $I_{i,h}$, $i=1,\dots,n$, are formal deformations of $I_i$, i.e. $I_{i,h}$ are $\mathbf{k}[[h]]$ submodules of $V[[h]]$, $I_{i,0}=I_i$, and $V^{\otimes 2}[[h]]=\bigoplus_i^n I_{i,h}$ is a direct sum decomposition. Then the deformations $Q_{m,h}$ and $M_h$ of the algebras $Q_m$ and $M$ are well defined in the natural way.

Theorem 1.2, which will be proven in the next section, states that all the deformations of those algebras are flat if the $\mathbf{k}[[h]]$ subalgebras $A_k(I_{1,h},\dots,I_{n,h})$ of $\mathrm{End}(V^{\otimes k})[[h]]$ analogous to those defined in the paragraph preceding Theorem 1.1 are splitting submodules for all $k$. These subalgebras are defined as follows. Take a $\mathbf{k}[[h]]$ linear operator $B_h$ acting on $V^{\otimes 2}[[h]]$ which has $n$ distinct eigenvalues with $I_{1,h},\dots,I_{n,h}$ as the corresponding eigensubmodules. Then $A_k(I_{1,h},\dots,I_{n,h})$ is generated by $B_{i,h}=\mathrm{id}\otimes\cdots\otimes B_h\otimes\cdots\otimes\mathrm{id}\in\mathrm{End}(V^{\otimes k})[[h]]$

where $B_h$ appears in positions $i, i+1$. Another set of generators for this subalgebra is given by the projections $P^m_{i,h}$ onto the $m^{th}$ eigensubmodule of $B_{i,h}$, thus $A_k(I_{1,h}, \dots, I_{n,h})$ really depends only on the decomposition of $V^{\otimes 2}[[h]]$, but not on choosing of $B_h$.

## 3. Semisimplicity and flatness

In this section we prove some known facts on finite dimensional semisimple algebras. Then we prove Theorems 1.1 and 1.2. In the next section we give some examples of application of these theorems.

A finite-dimensional representation $E$ of an algebra $A$ (or $A$-module) is called **simple** if there are no nontrivial invariant subspaces, and it is called **semisimple** if $E$ is isomorphic to a direct sum of simple representations. A finite-dimensional algebra is called semisimple if all its finite-dimensional representations are semisimple. An linear operator $B \in \mathrm{End}(E)$ is semisimple if the subalgebra of operators generated by it is semisimple. In general, we call a set of operators $\mathcal{F} \subset \mathrm{End}(E)$ semisimple if the subalgebra $A(\mathcal{F})$ generated by this family is semisimple.

As is known [Pie] an algebra $A$ will be semisimple if and only if its semisimple representations separate points, i.e. for any two elements $a, b \in A$ there exists a semisimple representation $\varphi$: $A \to \mathrm{End}(V)$ such that $\varphi(a) \neq \varphi(b)$. In particular, if $A$ is a subalgebra of $\mathrm{End}(E)$ and the space $E$ is a semisimple $A$-module then $A$ is semisimple. It follows from this that the following algebras are semisimple:

(a) $A(\varphi(\mathcal{G}))$ for a representation $\varphi$: $\mathcal{G} \to \mathrm{End}(E)$ of a semisimple or compact Lie algebra $\mathcal{G}$;

(b) $A(\varphi(G))$ for a representation $\varphi$: $G \to \mathrm{End}(E)$ of semisimple or compact Lie group $G$;

(c) $A(\varphi(\mathcal{F}))$ for any subset $\mathcal{F}$ of a compact Lie algebra or group and $\varphi$ is its representation.

Note that if $\varphi$ is a representation of a connected Lie group $G$ and $\psi$ is the corresponding representation of its Lie algebra $\mathcal{G}$ then the algebras $A(\varphi(G))$ and $A(\psi(\mathcal{G}))$ coincide.

PROPOSITION 3.1: *Let $E$ be a finite-dimensional vector space over* $\mathbf{k}$. *Suppose $B_1, \dots, B_m$ is a semisimple set of linear operators on $E$ and $\lambda_1, \dots, \lambda_m$ are elements from $K$. Denote $L = \sum_i \mathrm{Im}(B_i - \lambda_i)$, $K = \bigcap_i \mathrm{Ker}(B_i - \lambda_i)$. Then*

(a) *the subspaces $L$ and $K$ are invariant under all the $B_i$;*

(b) $E = L \oplus K$;

(c) $\mathrm{End}(E) = \sum_i \mathrm{Im}(\mathrm{ad}\, B_i) \oplus \bigcap_i \mathrm{Ker}(\mathrm{ad}\, B_i)$, *where the operator* $\mathrm{ad}\, B$, $B \in \mathrm{End}(E)$, *acts on* $\mathrm{End}(E)$ *as* $\mathrm{ad}\, B(X) = BX - XB$.

*Proof:* (a) The invariance of $K$ is obvious. Let $v = \sum_i (B_i - \lambda_i)u_i$. Then $B_j v = \sum_i B_j(B_i - \lambda_i)u_i = \sum_i (B_i - \lambda_i)B_j u_i + \sum_i [B_j, (B_i - \lambda_i)]u_i$, and (a) follows from the equality of commutators: $[B_j, (B_i - \lambda_i)] = [(B_j - \lambda_j), (B_i - \lambda_i)]$.

(b) Because of semisimplicity there exists an invariant subspace $P$ in $E$ complementary to $L$. If $v \in P$ then $(B_i - \lambda_i)v$ has to belong to both $L$ and $P$. Hence $(B_i - \lambda_i)v = 0$ for all $i$. It means that $v \in K$. So $P \subset K$ and $E = L + K$. Let now $T$ be the invariant subspace in $E$ complementary to $K$. It is clear that $L = \sum_i (B_i - \lambda_i)T$, so $L \subset T$ and, therefore, $L \cap K = 0$. This implies that $E = L \oplus K$.

(c) follows from the fact that $A(B_1, \ldots, B_m)$ is a reductive Lie algebra relative to commutator. The decomposition follows from the same argument as in (b). ∎

Now we consider deformations of semisimple algebras and their morphisms. In general, let $A_h$ be an algebra over $\mathbf{k}[[h]]$ which is a free $\mathbf{k}[[h]]$-module. Then $A_0 = A_h/hA_h$ is an algebra over $\mathbf{k}$, and we call $A_h$ a family of algebras, or a deformation of the algebra $A_0$. If $A_h'$ is another deformation of $A_0$ then a morphism of the deformations is a $\mathbf{k}[[h]]$-algebra morphism $A_h \to A_h'$ which is the identity for $h = 0$. The deformation is trivial if there exists an $\mathbf{k}[[h]]$-algebra isomorphism $A_h \to A_0[[h]] = A_0 \otimes_{\mathbf{k}} \mathbf{k}[[h]]$. We say that a subalgebra $B_h \subset A_h$ is **splitting** if it is a splitting $\mathbf{k}[[h]]$-submodule in $A_h$.

PROPOSITION 3.2: (a) *Let $A_h$ be a family of algebras. Suppose the algebra $A_0$ over $\mathbf{k}$ is semisimple. Then $A_h$ is isomorphic to $A_0[[h]]$ as $\mathbf{k}[[h]]$-algebra, i.e. the deformation is trivial.*

(b) *Let $\phi_h$: $A[[h]] \to B[[h]]$ be a morphism of $\mathbf{k}[[h]]$-algebras. It induces the morphism $\phi_0$: $A \to B$ of $\mathbf{k}$-algebras. Suppose $A$ is semisimple and $B$ is an arbitrary unital algebra. Then there exists an element $f_h \in B[[h]]$ such that $f_0 = 1$ and $\phi_h = \mathrm{Ad}(f_h)(\phi_0 \otimes \mathbf{1})$. Here $\mathrm{Ad}(b)c = bcb^{-1}$ and $\phi_0 \otimes \mathbf{1}$: $A \otimes_{\mathbf{k}} \mathbf{k}[[h]] \to B \otimes_{\mathbf{k}} \mathbf{k}[[h]]$ is the morphism of tensor products induced by $\phi_0$ and the identity morphism.*

*Proof:* The proposition follows from the fact that the Hochschild cohomology of any semisimple algebra is equal to zero [Pie] using the standard arguments [GGS]. More precisely, (a) follows from $H^2(A, A) = 0$ and (b) from $H^1(A, B) = 0$ where $B$ is considered as $A$-bimodule via the morphism $\phi_0$. ∎

Let families of algebras $A_h$ and vector spaces $V_h$ be given. Suppose the algebra $A_h$ acts on $V_h$, i.e. we are given a morphism of $\mathbf{k}[[h]]$-algebras $\varphi_h\colon A_h \to \mathrm{End}(V_h)$. Then $\varphi_h$ induces a morphism $\varphi_0\colon A_0 \to \mathrm{End}(V_0)$. On the other hand, any morphism $\psi\colon A_0 \to \mathrm{End}(V_0)$ generates in the trivial way the morphism $\psi\colon A_0[[h]] \to \mathrm{End}_{\mathbf{k}[[h]]}(V_0[[h]]) = \mathrm{End}_{\mathbf{k}}(V_0)[[h]]$, so as a consequence of the preceding proposition we get that if the algebra $A_0$ is semisimple then the morphisms $\varphi_h$ and $\varphi_0$ are conjugate.

The following proposition is an analog of Proposition 3.1 for the deformation situation.

PROPOSITION 3.3: *Let $E$ be a finite-dimensional vector space over $\mathbf{k}$. Suppose $B_1, \dots, B_m$ is a semisimple set of semisimple linear operators on $E$, $\lambda_1, \dots, \lambda_m$ are elements from $\mathbf{k}$. Let $B_{ih} \in \mathrm{End}(E)[[h]]$ and $\lambda_{ih} \in \mathbf{k}[[h]]$ be deformations of $B_i$ and $\lambda_i$ ($i = 1, \dots, m$) such that*

(i) *all the subalgebras $A_{ih} = A(B_{ih})$ are splitting submodules;*

(ii) *the subalgebra $A_h = A(B_{1h}, \dots, B_{mh})$ is a splitting submodule;*

(iii) *all the submodules $K_{ih} = \mathrm{Ker}(B_{ih} - \lambda_{ih})$ are splitting ones.*

*Denote $L_h = \sum_i \mathrm{Im}(B_{ih} - \lambda_{ih})$, $K_h = \bigcap_i \mathrm{Ker}(B_{ih} - \lambda_{ih})$. Then*

(a) *the submodules $L_h$ and $K_h$ are invariant under all $B_{ih}$;*

(b) *$E[[h]] = L_h \oplus K_h$. In particular, $L_h$ and $K_h$ are splitting submodules;*

(c) *$\mathrm{End}(E)[[h]] = \sum_i \mathrm{Im}(\mathrm{ad}\, B_{ih}) \oplus \bigcap_i \mathrm{Ker}(\mathrm{ad}\, B_{ih})$.*

*Proof:* The invariance of $L_h$ and $K_h$ can be proven as in Proposition 3.1. First, suppose that the algebra $A_h$ generated by $B_{ih}$ has the form $A_0[[h]]$ there $A_0 = A(B_1, \dots, B_m)$. Denote $L_0 = \sum_i \mathrm{Im}(B_i - \lambda_i)$, $K_0 = \bigcap_i \mathrm{Ker}(B_i - \lambda_i)$. If $K_0 = 0$ then $L_0 = E$ by Proposition 3.1, and (b) is obvious from the fact that $\dim L_0 \le \dim L_{h\neq 0}$. Suppose $K_0 \neq 0$. Then, since $K_0$ is an eigensubspace for all elements from $A_0$, the elements $\lambda_i$ define a character, i.e. an algebra homomorphism, $\chi_0\colon A_0 \to \mathbf{k}$ by $\chi_0(B_i) = \lambda_i$. In the same way the element $\lambda_{ih}$ and submodule $K_{ih}$ define a character $\rho_{ih}\colon A_{ih} \to \mathbf{k}[[h]]$ for all $i = 1, \dots, m$. Consider the morphism $\chi_h = \chi_0 \otimes \mathbb{1}\colon A_0[[h]] \to \mathbf{k}[[h]]$. Since the algebras $A(B_i)$ are semisimple and the restriction of $\chi_0$ onto $A(B_i)$ coincides with $\rho_{i0}$ for all $i$, it follows from (i) and Proposition 3.2(b) that $\chi_h = \rho_{ih}$ on $A_{ih}$ for all $i$. This implies that

$$L_h = \sum_{B \in A_h} \mathrm{Im}(B - \chi_h(B)), \quad K_h = \bigcap_{B \in A_h} \mathrm{Ker}(B - \chi_h(B)).$$

Taking into account that $\chi_h = \chi_0 \otimes \mathbb{1}$ we get that $L_h = L_0[[h]]$ and $K_h = K_0[[h]]$, which proves (b).

(c) follows from the fact that $\bigcap_i \operatorname{Ker}(\operatorname{ad} B_{ih})$ coincides with the commutator of the subalgebra $A(B_1, \ldots, B_m)[[h]]$ in $\operatorname{End}(E)[[h]]$ and from Proposition 3.1, which proves the proposition in the case $A_h = A_0[[h]]$.

Suppose now that $A_h$ is arbitrary. Then, by (ii) and Proposition 3.2 there exists an element $f_h \in \operatorname{End}(V)[[h]]$ such that $f_h A_h f_h^{-1} = A_0[[h]]$. Constructing the spaces $L_h^{'} = \sum_i \operatorname{Im}(B_{ih}^{'} - \lambda_{ih})$, $K_h^{'} = \bigcap_i \operatorname{Ker}(B_{ih}^{'} - \lambda_{ih})$ for $B_{ih}^{'} = f_h B_{ih} f_h^{-1}$ we obtain that the modules $L_h = f_h^{-1} L_h^{'}$ and $K_h = f_h^{-1} K_h^{'}$ satisfy (b) of the proposition. Applying the same conjugation by $f_h$ we obtain (c). ∎

Let $B_1, \ldots, B_m$ is a semisimple set of semisimple operators in a vector space $E$. We say that deformations of these operators, $B_{1h}, \ldots, B_{mh}$, form a flat deformation of the set if the conditions (i) and (ii) from Proposition 3.3 hold.

*Remark 3.1:* It is clear that if a semisimple operator $B$ on $E$ and its flat deformation $B_h$ are given then for any eigenvalue $\lambda$ of $B$ its deformation $\lambda_h$ is uniquely defined. Furthermore, $K_h = \operatorname{Ker}(B_h - \lambda_h)$ and $L_h = \operatorname{Im}(B_h - \lambda_h)$ form deformations of the subspaces $K = \operatorname{Ker}(B - \lambda)$ and $L = \operatorname{Im}(B - \lambda)$ in $E$. Indeed, $\lambda$ defines a character $\chi$: $A(B) \to \mathbf{k}$, $\chi(B) = \lambda$, which, by Proposition 3.2, has the unique extension $\chi_h$: $A(B_h) \to \mathbf{k}[[h]]$. Then, $\lambda_h = \chi_h(B_h)$. So, it follows from this that if $B_i$, $\lambda_i$, $i = 1, \ldots, m$, is a set of semisimple operators on $E$ with fixed eigenvalues and $B_{ih}$ is a flat deformation of the set, then the deformations of the eigenvalues, $\lambda_{ih}$, exist and are uniquely defined such that the condition (iii) of Proposition 3.3 is satisfied and, therefore, for these $\lambda_{ih}$ the proposition holds.

Now we return to the setting of the end of Section 2.

PROPOSITION 3.4: *Let $V$ be a finite-dimensional vector space over $\mathbf{k}$, $B$ a linear operator on $V^{\otimes 2}$, and $B_h$ a deformation of $B$. Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $B$ and $I_1, \ldots I_n$ are the corresponding eigensubspaces. Suppose that the subalgebras $A_k(B)$ in $\operatorname{End}(V^{\otimes k})$ are semisimple and the subalgebras $A_k(B_h)$ in $\operatorname{End}(V^{\otimes k})[[h]]$ are splitting submodules for all $k \geq 2$. Then deformations of the eigenvalues, $\lambda_{ih}$, and eigensubspaces, $I_{ih}$, $i = 1, \ldots, n$, are uniquely defined and*

- (a) *the pairs of submodules $(I_{m,h}, J_{(m).h})$, $m = 1, \ldots, n$, are well situated, where $J_{(m),h} = \bigoplus_{i \neq m} I_{ih}$;*
- (b) *the quadratic algebras $(V[[h]], J_{(m),h})$ form flat deformations of the quadratic algebras $(V, J_{(m)})$ for all $m$;*
- (c) *the quantum semigroup $M(V[[h]]; I_{1h}, \ldots, I_{nh})$ is a flat deformation of the quantum semigroup $M(V; I_1, \ldots, I_n)$.*

*Proof:* It is clear that $B_{1h}, \ldots, B_{(k-1)h}$ form a flat set of semisimple operators in $\mathrm{End}(V^{\otimes k})$ for all $k$. The deformations of the eigenvalues, $\lambda_{ih}$, and eigensubspaces, $I_{ih}$, $i = 1, \ldots, n$, are uniquely defined by Remark 3.1. Noting that $I_m^{(k)} = \bigcap_{i=1}^{k-1} \mathrm{Ker}(B_i - \lambda_m)$ and $J_m^k = \sum_{i=1}^{k-1} \mathrm{Im}(B_i - \lambda_m)$ and applying Proposition 3.3 we obtain assertions (a) and (b).

For proving (c) we note that the $k$-th homogeneous component of the quantum semigroup $M(V[[h]]; I_{1h}, \ldots, I_{nh})$ is equal to $\mathrm{End}(V^{\otimes k})[[h]]/\sum_i \mathrm{Im}(\mathrm{ad}\, B_{ih})$. But by the assertion (c) of Proposition 3.3, $\sum_i \mathrm{Im}(\mathrm{ad}\, B_{ih})$ is a splitting submodule in $\mathrm{End}(V^{\otimes k})[[h]]$, so that component is a free module. ∎

*Remark 3.2:* One can consider the case when the variable $h$ runs through a complex or real analytic manifold $X$, the subspaces $I_{ih}$ depend on $h$ analytically, and one has the decomposition $V^{\otimes 2} = \bigoplus_i^n I_{ih}$ at any point $h \in X$. Suppose $\mathrm{dim} A_k(I_{1h}, \ldots, I_{nh})$ does not depend on $h$ (this condition replaces the condition of splitting of the subalgebra in the formal case). Suppose $A_k(I_{1h_0}, \ldots, I_{nh_0})$ is semisimple at one point $h_0 \in X$. Then there exists an analytic subset $Y_k \subset X$ such that for $h \in X \backslash Y_k$ the algebra $A_k(I_{1h}, \ldots, I_{nh})$ is semisimple and isomorphic to $A_k(I_{1h_0}, \ldots, I_{nh_0})$ (cf. Proposition 3.2 (a)). Following the arguments of this section one can prove that for $h \in X \backslash \cup Y_k$ all the quadratic algebras $Q(V, J_{(m),h})$ have constant dimension in all homogeneous components. The same is true for the corresponding quantum semigroups.

*Proof of Theorems 1.1 and 1.2:* Theorem 1.2 follows immediately from Proposition 3.4 if to note that $I_{1,h}, \ldots, I_{n,h}$ can be realized as eigensubspaces of a family of semisimple operators, $B_h \in \mathrm{End}(V^{\otimes 2}[[h]]$, therefore $A_k(I_{1,h}, \ldots, I_{n,h}) = A_k(B_h)$. Theorem 1.1 follows from Remark 3.2.

## 4. Applications

*Example 1:* Let $V$ be a finite-dimensional vector space over $\mathbf{k}$ ($\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$). Let $S$ be an invertible linear operator on $V \otimes V$ with two eigenvalues $\lambda$ and $\mu$ satisfying the braid relation (or quantum Yang–Baxter equation)

$$S_1 S_2 S_1 = S_2 S_1 S_2 \tag{1}$$

on $V^{\otimes 3}$. In this case the subalgebras $A_k(S) \subset V^{\otimes k}$ are images of the Hecke algebras. The Hecke algebra $H_k(\lambda, \mu)$ is defined as the quotient algebra of the free algebra $T(x_1, \ldots, x_{k-1})$ of $k-1$ variables by the relations

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}, \quad x_i x_j = x_j x_i \quad \text{for } | i - j | \geq 2, \tag{2}$$

(3) $$(x_i - \lambda)(x_i - \mu) = 0.$$

It is known, [Co], that for almost all pairs $(\lambda, \mu)$ (excepting an closed algebraic subset) this algebra is semisimple and isomorphic to the group algebra, $H_k$, of the symmetric group (the case $\lambda = 1$, $\mu = -1$). Moreover, in a neighborhood of each point $(\lambda_0, \mu_0)$ this isomorphism can be chosen analytically dependent on $\lambda$ and $\mu$. We suppose that the eigenvalues, $\lambda$ and $\mu$, of $S$ correspond to the semisimple Hecke algebra. In this case $S$ is called a Hecke symmetry. Gurevich [Gu] considered the case in details. Now we consider deformations of the Hecke symmetry.

Let $S_h$ be a deformation of the operator $S$ satisfying the relations

(4) $$S_{h1}S_{h2}S_{h1} = S_{h2}S_{h1}S_{h2},$$

(5) $$(S_h - \lambda_h)(S_h - \mu_h) = 0,$$

where $\lambda_h$ and $\mu_h$ are deformations of $\lambda$ and $\mu$. Let us prove that in this case the subalgebras $A_k(S_h)$ for all $k \geq 2$, are splitting. Indeed, due to relations (4) and (5) there exists an algebra homomorphism $\phi_h$: $H_k[[h]] \to \mathrm{End}(V^{\otimes k})[[h]]$ such that $\mathrm{Im}(\phi_h) = A_k(S_h)$. Using Proposition 3.2 we conclude that the algebra $A_k(S_h)$ is isomorphic to $A_k(S)[[h]]$ and, therefore, splitting.

*Example 2:* We obtain the same result if $S$ satisfies the Birman–Wenzl relations:

(a) the braid relation (1);
(b) the cubic relation $(S - \lambda)(S - \mu)(S - \nu) = 0$ for $\lambda, \mu, \nu \neq 0$;
(c) $P_1S_2P_1 = aP_1$, where $P = (S - \lambda)(S - \mu)$ and $a$ is a constant;
(d) $P_1P_2P_1 = bP_1$, where $b$ is a constant.

It follows from (b) that $S$ has three eigenvalues and eigensubspaces.

In this case the subalgebras $A_k(S) \subset V^{\otimes k}$ are images of the Birman–Wenzl (BW) algebras $BW_k$ [BW]. The algebra $BW_k$ is defined as the quotient algebra of the free algebra $T(x_1, \ldots, x_{k-1})$ of $k - 1$ variables by the relations (2) and

$$(x_i - \lambda)(x_i - \mu)(x_i - \nu) = 0,$$
$$p_i x_{i\pm1} p_i = a p_i,$$
$$p_i p_{i\pm1} p_i = b p_i,$$

where $p_i = (x_i - \lambda)(x_i - \mu)$. One can show that the constants $a$ and $b$ are uniquely defined, and $a = \lambda\mu(\lambda + \mu)$, $b = (\lambda + \mu)^2\nu^2$. Note that in [BW] BW algebras are defined by eleven relations; see [Ke] where it is proven that the algebra $BW_k$ can be defined as above.

It is known, [BW], that for almost all triples $\lambda$, $\mu$, and $\nu$ this algebra is semisimple and analytically depended on $\lambda$, $\mu$, $\nu$. We suppose that $\lambda$, $\mu$ and $\nu$ form such a triple. In this case $S$ satisfying the relations (a)–(d) is called a Birman–Wenzl symmetry. So, in the case of BW symmetry algebras $A_k(S)$ are semisimple as well.

If by a deformation of $S$ the relations (a)–(d) hold (with deformed eigenvalues $\lambda$, $\mu$, $\nu$) we say that this is a deformation of Birman–Wenzl symmetry. Using the same arguments as in (1) we obtain that by a deformation $S_h$ of the BW symmetry $S$ the algebras $A_k(S_h)$ are splitting.

Applying Proposition 3.4 we obtain

PROPOSITION 4.1: *Let $S$ be a Hecke (resp. BW) symmetry on the space $V$, $I$ its eigensubspace in $V \otimes V$, and $J$ is the sum of other eigensubspaces. Suppose $S_h$ is a deformation of the Hecke (resp. BW) symmetry. Then the deformation defines a flat deformation $Q(V[[h]], J_h)$ of the quadratic algebra $Q(V, J)$ and the pair $(I_h, J_h)$ is well situated. Moreover, the deformation of the quantum semigroup corresponding to the eigensubspaces of $S$ is flat.*

This proposition for the case $S = \sigma$ is proven in [GGS1]. Note that Gurevich proved in [Gu] that in case of Hecke symmetry the algebra $Q(V, I)$ is Koszul. He also constructed Hecke symmetries with nonclassical dimensions of homogeneous components of $(V, I)$.

In particular, deformations of the Hecke and BW symmetries appears in [FRT] by construction of the quantum analogs (deformations) of the classical Lie groups. Namely, the Hecke symmetry corresponds to the case of general linear group, while the BW symmetry corresponds to the orthogonal and symplectic cases.

*Example 3:* Let $U_h(\mathcal{G})$ be the Drinfeld–Jimbo quantized universal enveloping algebra (DJ quantum group), for a semisimple Lie algebra $\mathcal{G}$ over $\mathbf{k} = \mathbb{C}$. Let $R \in U_h(\mathcal{G}) \otimes U_h(\mathcal{G})$ be the corresponding quantum R-matrix. Suppose, a representation $V_h$ of $U_h(\mathcal{G})$ is given, which is a deformation of the finite dimensional representation $V$ of $U(\mathcal{G})$, so $V_h$ is isomorphic to $V[[h]]$ as $\mathbf{k}[[h]]$-module, and the representation can be presented as a homomorphism $\rho$: $U_h(\mathcal{G}) \to \mathrm{End}(V)[[h]]$. Consider the operator $S_h = \sigma R_h$ where $R_h = (\rho \otimes \rho)(R) \subset \mathrm{End}(V)^{\otimes 2}[[h]]$ and $\sigma$ is the standard permutation. It is known that $S_h$ is a Yang–Baxter operator, i.e. satisfying the braid relation (1). But it is not necessarily a flat deformation of semisimple operator, because at $h = 0$ the operator $S_0$ is equal to $\sigma$, so has two eigenvalues, $\pm 1$, while at the general point $h \neq 0$ it is semisimple but may have more than two eigenvalues, $\lambda_{ih}$, $i = 1, \ldots, n$, such that $\lambda_{i0} = \pm 1$.

Nevertheless, there is the decomposition $V[[h]] = \bigoplus_i I_{ih}$ where $I_{ih}$ are eigen-submodules of $S_h$ corresponding to $\lambda_{ih}$, and, therefore, all $I_{ih}$ are splitting. We will prove that also in this setting the decomposition defines flat deformations of quadratic algebras, $Q(V[[h]], J_{((m),h)})$, $J_{(m),h} = \bigoplus_{i\neq m} I_{ih}$, and of the corresponding quantum semigroup, $M(V[[h]]; I_{1h}, \dots, I_{nh})$. For this, according to Theorem 1.2 we need to show that the algebras $A_k(I_1, \dots, I_n)$ are semisimple, $I_i = I_{i0}$, and $A_k(I_{1h}, \dots, I_{nh})$ are splitting for $k \geq 2$.

We recall some results of Drinfeld from [Dr1] and [Dr2]. Additional structures on the category $Rep_A$ of representations of an associative algebra $A$ and morphisms of these structures can be given by the additional structures on the algebra $A$ itself. Thus, the structure of quasitensor monoidal category on $Rep_A$ can be given with the help of an algebra homomorphism $A \to A \otimes A$ (comultiplication), an element $\Phi \in A^{\otimes 3}$ (associativity constraint), and R-matrix $R \in A^{\otimes 2}$ (commutativity constraint), satisfying the certain conditions. A morphism of such two structures can be given by an element $F \in A^{\otimes 2}$. Drinfeld defined such a structure on $A = U(\mathcal{G})[[h]]$ for any semisimple Lie algebra $\mathcal{G}$ with the usual comultiplication $\Delta$ but nontrivial $R$ and $\Phi$. He then proved that the corresponding quasitensor category is isomomorphic by some $F_h$ to the category of representations of the Drinfeld–Jimbo quantum group $U_h(\mathcal{G})$ which coincides with $U(\mathcal{G})[[h]]$ as an algebra but has a noncommutative comultiplication $\tilde{\Delta}$. We denote the corresponding quasitensor categories by $\mathcal{C}$ and $\tilde{\mathcal{C}}$, respectively. We keep the notations $\mathcal{R}_h$ and $\Phi_h$ for the R-matrix and the associativity constraint in the category $\mathcal{C}$, while $R_h$ denote the R-matrix for $\tilde{\mathcal{C}}$. Further, Drinfeld proved that $\mathcal{R}_h$ and $\Phi_h$ may be chosen as $\mathcal{R}_h = e^{h\mathbf{t}}$ where $\mathbf{t} \in \mathcal{G} \otimes \mathcal{G}$ is the split Casimir, and $\Phi_h = e^{L(h\mathbf{t}_1, h\mathbf{t}_2)} \in U(\mathcal{G})^{\otimes 3}[[h]]$ where $L(h\mathbf{t}_1, h\mathbf{t}_2)$ is a Lie expression of $\mathbf{t}_1 = \mathbf{t} \otimes 1$ and $\mathbf{t}_2 = 1 \otimes \mathbf{t}$. The element $F_h \in U(\mathcal{G})^{\otimes 2}[[h]]$ is congruent to $1 \otimes 1$ modulo $h$ and satisfies the equation

$$(F_h \otimes 1) \cdot (\Delta \otimes \mathrm{id})(F_h) = (1 \otimes F_h) \cdot (\mathrm{id} \otimes \Delta)(F_h) \cdot \Phi_h. \tag{6}$$

According to this, the commutativity constraints in the categories $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are given by the elements

$$\mathcal{S}_h = \sigma \mathcal{R}_h = \sigma e^{h\mathbf{t}} \quad \text{and} \quad S_h = F_h \mathcal{S}_h (F_h)^{-1} = F_h \sigma e^{h\mathbf{t}} (F_h)^{-1}, \tag{7}$$

respectively.

Let us come back to the setting from the beginning of (4). Using (7) we obtain that the eigenvalues of the operators $\mathcal{S}_h$ and $S_h$ acting on $(V \otimes V)[[h]]$ are $\lambda_{ih} = \pm e^{h\lambda_i}$ where $\lambda_i$ are the eigenvalues of $\mathbf{t}$ on the same space.

Now we transfer the setting to the category $\mathcal{C}$, i.e. we consider $V[[h]]$ as an object of $\mathcal{C}$. Instead of $S_h$ we consider $\mathcal{S}_h$, but the tensor products depend on the placement of parentheses and the connection between two bracketing, $\phi_h\colon (V^{\otimes k})' \to (V^{\otimes k})''$, of the $k$-fold tensor product is generated by the operator

$$(8) \qquad \Phi_h\colon ((V\otimes V)\otimes V)[[h]] \to (V\otimes (V\otimes V))[[h]]$$

and looks like an expression $\phi_h \in \mathrm{End}(V^{\otimes k})[[h]]$ depending on the elements $\mathbf{t}_{i,j} = 1\otimes\cdots\otimes\mathbf{t}_{(1)}\otimes\cdots\otimes\mathbf{t}_{(2)}\otimes\cdots\otimes 1$ ($\mathbf{t}_{(1)}$ and $\mathbf{t}_{(2)}$ at the places $i$ and $j$, $\mathbf{t}=\mathbf{t}_{(1)}\otimes\mathbf{t}_{(2)}$ in the Sweedler notations). It is easy to see that the eigensubmodules of the operator $\mathcal{S}_h$ have the form $\mathcal{I}_i[[h]]$ where $\mathcal{I}_i$ are the common eigensubmodules of $\mathbf{t}$ and $\sigma$. Denote by $A_k(\sigma,\mathbf{t})$ the subalgebra in $\mathrm{End}(V^{\otimes k})$ generated by the elements $\sigma_{i,i+1}$ and $\mathbf{t}_{i,i+1}$. So we get that the algebra $A_2(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]])$ is equal to $A_2(\sigma,\mathbf{t})[[h]]$. The algebras $A_k{}'(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]])$ and $A_k{}''(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]])$ for two bracketings $(V^{\otimes k})'$ and $(V^{\otimes k})''$ are connected by $A_k{}'(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]]) = \phi_h^{-1}A_k{}''(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]])\phi_h$. Using the relations of the type $\mathbf{t}_{1,3}=\sigma_{2,3}\mathbf{t}_{1,2}\sigma_{2,3}$ we conclude that $\phi_h\in A_k(\sigma,\mathbf{t})[[h]]$, and using the fact that $\phi_h$ is congruent to $1\otimes 1\otimes 1$ modulo $h^2$, we conclude by induction on $k$ that $A_k{}'(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]]) = A_k{}''(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]]) = A_k(\sigma,\mathbf{t})[[h]]$.

Passing to the category $\tilde{\mathcal{C}}$ we obtain that

$$A_k(I_{1h},\dots,I_{nh}) = f_h{}'A_k{}'(\mathcal{I}_1[[h]],\dots,\mathcal{I}_n[[h]])(f_h{}')^{-1} = f_h{}'A_k(\sigma,\mathbf{t})[[h]](f_h{}')^{-1}$$

where $f_h{}'$ is the composition (depending on the bracketing $'$) of a number of $F_h$ with $\Delta$ applied appropriate factors. (We would obtain the same result applying to $A_k(\sigma,\mathbf{t})$ the element $f_h{}''$ related to the bracketing $''$.) So, we have proved that $A_k(I_{1h},\dots,I_{nh})$ is splitting.

For $h=0$ this algebra is equal to $A_k(\sigma,\mathbf{t})$. Let us prove that it is a semisimple algebra. Indeed, $\mathbf{t}$ may be presented as $\mathbf{t}=\sum_i d_i\otimes d_i$ where $d_i$ form an orthogonal (with respect to the Killing form) basis in the maximal compact subalgebra $\mathcal{K}$ of $\mathcal{G}$. Hence, there exists a Hermitian metric on $V$ invariant under action of $\mathcal{K}$. This metric induces naturally the metric on $V^{\otimes k}$ which will be invariant under the operators $\mathbf{t}_{i,j}$ and $\sigma$. So, these operators are unitary ones, therefore the algebra $A_k(\sigma,\mathbf{t})$ generated by them is semisimple.

Thus, we obtain

PROPOSITION 4.2: *Let $S_h$ be the Yang–Baxter operator on a space $V$, which is obtained by the representation of the Drinfeld–Jimbo quantum group. Then $S_h$ defines the decomposition $V^{\otimes 2}=\bigoplus_{i=1}^n I_{ih}$ into eigensubmodules. Let $J_{(m),h}=\bigoplus_{i\neq m} I_{ih}$. Then $Q(V[[h]],J_{(m),h})$ are flat deformations of the quadratc algebras* *and the pairs* $(I_{m,h}, J_{(m),h})$ *are well situated. Moreover, the deformation of the quantum semigroup* $M(V[[h]]; I_{1h}, \ldots, I_{nh})$ *is flat.*

Another proof of this proposition is contained in [DS].

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